

NUMERICAL METHODS

Chapter 4: Iterative Methods for Scalar Nonlinear Equations

Assoc. Prof. Dr. Annie Gorgey

Email: annie_gorgey@fsmt.upsi.edu.my

Department of Mathematics

Faculty of Science and Mathematics

Sultan Idris Education University, Malaysia

Learning outcomes

- ★ To identify the types of methods suitable in solving scalar nonlinear equations
- ★ To find the root of a nonlinear equations using Newton-Raphson, Secant and Bisection methods.
- ★ To approximate the solution of a nonlinear scalar equation by hand using the Newton-Raphson, Secant and Bisection methods.
- ★ To study the difference between local and global convergence.
- ★ To estimate the rate of convergence of an iterative scheme;

Contents

- 1 *Intermediate Value Theorem (IVT)*
- 2 *Scalar Nonlinear Equations*
 - Roots of nonlinear equations
- 3 *Iterative methods*
- 4 *Convergence*

Intermediate Value Theorem (IVT)

Theorem

Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and let L be a real number that lies in $[a, b]$ with $f(a) \leq L \leq f(b)$, then there is one c with $a < c < b$ such that

$$L = f(c).$$

This IVP is important to determine whether there exists a point below or above on a given particular line. It is also used to analyze the continuity of a function that whether it is continuous or not.

Example

Show that there is some x with $0 \leq x \leq 2$ such that $x^2 + \cos(\pi x) = 4$.

Solution

To apply the intermediate value theorem (IVT), let $f(x) = x^2 + \cos(\pi x)$.
By the IVT

$$f(0) = (0)^2 + \cos(0) = 1.$$

$$f(2) = (2)^2 + \cos(2\pi) = 5.$$

Since $f(0) = 1 \leq 4 \leq 5 = f(2)$ and since f is continuous, the IVT implies that $f(x) = 4$ for some x in the closed interval $[0, 2]$. This x cannot be one of the endpoints, so $0 < x < 2$.

Mean Value theorem (MVT)

Another important theorem is the Mean Value Theorem (MVT) that is given next.

Theorem

Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and it is differentiable for (a, b) . Then there is at least one point c in (a, b) for which

$$f(b) - f(a) = f'(c)(b - a).$$

The MVT describes a relationship between average rate of change and instantaneous rate of change. Geometrically, the MVT describes a relationship between the slope of a secant line and the slope of the tangent line.

Scalar Nonlinear Equations

Linear or quadratic equations are always easier to get the solution if compared with a scalar nonlinear equations. For example if

$$f(x) = x^2 + 2x - 3,$$

Solution for x simply means finding the value of x such that $f(x) = 0$. The solution to this can be easily obtained by using the quadratic formula or by completing the square.

But how would you find the value of x such that $g(x) = 0$, where

$$g(x) = x^2 + 3x - e^x.$$

Both $f(x)$ and $g(x)$ given above are nonlinear functions. But they differ in one important way. For $f(x) = 0$, the value of x can be found analytically but not for $g(x)$.

Nonlinear equation

Definition

A function is said to be nonlinear if the independent variable such as x occurs other than as the zeroth or first power.

Problem: Given $f(x)$, find a root (solution) of $f'(x) = 0$.

- If f is a polynomial with degree 4 or less, formulas for roots exist.
- If f is a polynomial with degree larger than 4, no finite formula exists (proved by Abel in 1824).
- If f is a general nonlinear function, no formula exists.

Since no formula exists for roots of $f(x) = 0$, iterative methods can be used to compute approximate roots.

The iterative methods discussed in this chapter are used to approximate solution of equations where the exact or analytic solution are hard to obtain.

Iterative methods construct a sequence of numbers $x^{(1)}, x^{(2)}, \dots, x^{(k)}, x^{(k+1)}, \dots$ that converge to a root of $f(x) = 0$.

To find such approximations for a scalar nonlinear equation, the iterative methods that can be used are

- 1 Bisection method
- 2 Secant method, and
- 3 Newton (Raphson) method

Three major issues implementation of an iterative method are as follows:

- 1 Where to start the iteration?
- 2 Does the iteration converge, and how fast?
- 3 When to terminate the iteration?

NOTE: The main idea of this chapter is to find the root of a nonlinear equation such that when $f(x) = 0$, $g(x) = 0$ etc. In a simple words, equating an equation to zero simply means we are finding the root or the solution of x .

Roots of nonlinear equations

Example

Find all the roots of $x^2 + 2x - e^x = 0$ in the interval $-3 \leq x \leq 3$?

Solution

Graphically, its easier to find the solution by plotting the graph of $f(x) = x^2 + 2x - e^x$. The roots can be found by finding the solution at $f(x) = 0$ or $y = 0$.

According to Figure 1 at $y = 0$, the roots are $x = -2.2$, $x = 0.8$ and $x = 2.25$.

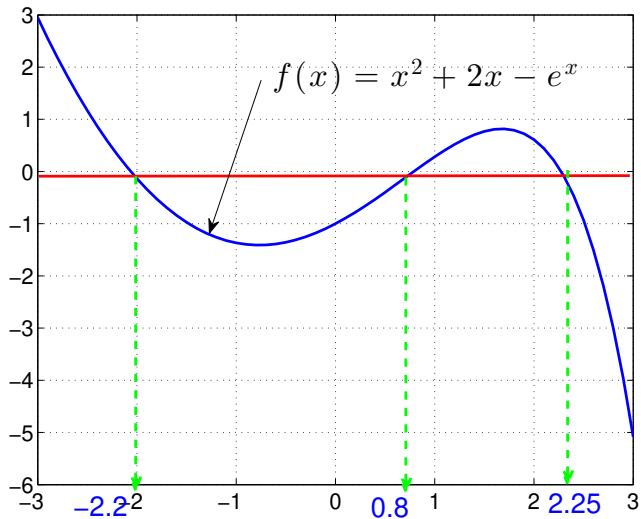


Figure: Graph of $f(x) = x^2 + 2x - e^x$

Locating the roots

The roots of the nonlinear equations can be shown graphically. However, to locate interval of root without plotting graph, one need to apply **Intermediate Value Theorem**.

Theorem

Let $f(x) = 0$ is a nonlinear equation. If $f(a)f(b) < 0$, then it is guaranteed that exists at least a root in the interval $[a, b]$.

Why is it important to locate/find the roots of nonlinear equations

All nonlinear equations can only be solution iteratively. To start an iterative procedure an approximate root need to be guessed. The better choice of the initial root, the more chances, to find the right root in shorter time.

Once the roots had been located, that roots can be used as the initial estimates.

The first iterative that will be studied is the **Bisection method**.

Class Exercise

Show whether there is a root for the given equations at the given interval using Intermediate Value Theorem.

❶ $4x^3 + 3x^2 + 2x + 11 = 0, \quad [-2, -1].$

❷ $3x^3 + 2x^2 + x + 10 = 0, \quad [-1.8, -1.5].$

Class Exercise

Locate all the roots of the given functions by plotting the graph of that function at any intervals.

- 1 $f(x) = x - x \ln(x)$.
- 2 $f(x) = \cos(x) - 2x + 3$.
- 3 $f(x) = x^3 - 10x + 10$.
- 4 $f(x) = x \cos(x) - 1$.

Bisection Method

Definition

Two distinct values of x are said to form a bracket for a root of $f(x) = 0$ if the values of $f(x)$ at the two points are of opposite sign.

For example, if $f(x) = x^2 - 1$, the values 0 and 2 form a bracket because $f(0)$ is negative and $f(2)$ is positive.

Explanation:

$$f(x) = x^2 - 1.$$

When $x = 0$ such that $f(0)$,

$$f(0) = -1, \text{ A negative value is obtained}$$

When $x = 2$, such that $f(2)$,

$$f(2) = 3, \text{ A positive value is obtained}$$

Because two values of x are distinct and are of opposite signs negatives and positives, the values of x are said to form a bracket for the root of $f(x) = 0$.

Step by Step Guide to Bisection method

- 1 Find two values of $[a, b]$ which satisfies the Intermediate Value Theorem

$$f(a)f(b) < 0.$$

- 2 Find the value of c using

$$c = \frac{a + b}{2}.$$

Step by Step Guide to Bisection method

➊ Having obtained the values of a , c and b , find the values of $f(a)$, $f(b)$ and $f(c)$.

➋ To go to the next iteration, check the following:

If $f(a)f(c) < 0$, then set

$$a = a.$$

$$b = c.$$

else set

$$a = c.$$

$$b = b.$$

➌ Repeat the process until $f(c) = 0$ or $|f(c)| < \epsilon$ or $|b_k - a_k| < \epsilon$, where ϵ can be a very small number.

NOTE:

If the stopping criteria is not specified in a particular problem, then from those three criteria, whichever comes first will be employed.

Bisection Method

Example

Apply five iterations of Bisection to $f(x) = x^2 - e^{-x}$. Use the bracket $[0, 1]$.

Solution

The solutions are given by following the step-by step guide to Bisection method.

- 1 Find two values of $[a, b]$ which satisfies the Intermediate Value Theorem.

The two values of a and b are given as $a = 0$ and $b = 1$.

- 2 Find the value of c using $c = \frac{a + b}{2}$.

Given $a = 0$ and $b = 1$. Thus,

$$c = \frac{a + b}{2} = \frac{0 + 1}{2} = 0.5.$$

Solution

- ③ Having obtained the values of a , c and b , find the values of $f(a)$, $f(b)$ and $f(c)$.

$$f(x) = x^2 - e^{(-x)}.$$

$$f(a) = f(0) = (0)^2 - e^{-0} = -1.$$

$$f(b) = f(1) = (1)^2 - e^{-1} = 0.6321.$$

$$f(c) = f(0.5) = (0.5)^2 - e^{-0.5} = -0.3565.$$

Solution

- ④ To go to the next iteration, check the following:
If $f(a)f(c) < 0$, then set

$$a = c.$$

$$b = b.$$

else set

$$a = a.$$

$$b = c.$$

Solution

- ⑤ Finding $f(a)f(c) < 0$ gives

$$f(a)f(c) = f(0)f(0.5) = (-1)(-0.3565) = 0.3565.$$

Since $f(a)f(c) > 0$, set $a = c$ and $b = b$ such that

$$a = 0.5.$$

$$b = 1.$$

- ⑥ Continue finding the values of $f(0.5)$, $f(1)$, $c = (a + b)/2$ and $f(c)$. Check for $f(a)f(c) < 0$. The process is repeated until $f(c) = 0$ or $|f(c)| < \epsilon$ or $|b_k - a_k| < \epsilon$.

This process can be made simpler by producing the answers in the table (Refer to Table 1).

Table: Solution to Example 10 using Bisection method.

k	a	b	c	$f(a)$	$f(b)$	$f(c)$
1	0.0000	1.0000	0.5000	-1.0000	0.6321	-0.3565
2	0.5000	1.0000	0.7500	-0.3565	0.6321	0.0901
3	0.5000	0.7500	0.6250	-0.3565	0.0901	-0.1446
4	0.6250	0.7500	0.6875	-0.1446	0.0901	-0.0302
5	0.6875	0.7500	0.7188	-0.0302	0.0901	0.0292

Therefore, the root lies in the interval $[0.69, 0.72]$ where the root is close to 0.72.

The results shown in Table 1 is just for six iterations. In this case, the stopping criteria $f(c) = 0$ or $|f(c)| < \epsilon$ or $|b_k - a_k| < \epsilon$ has not been implemented.

Example

Find the root of

$$f(x) = x + \ln(x),$$

using Bisection method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[0.2, 1]$.

Solution

Given $a = 0.2$ and $b = 1$. The value of c can be easily obtained from the formula.

$$c = \frac{a + b}{2} = \frac{0.2 + 1}{2} = 0.6.$$

Having obtained c , next is to find the values of $f(a)$, $f(b)$ and $f(c)$.

$$f(x) = x + \ln(x).$$

$$f(a) = f(0.2) = 0.2 + \ln(0.2) = -1.4094.$$

$$f(b) = f(1) = 1 + \ln(1) = 1.0000.$$

$$f(c) = f(0.6) = 0.6 + \ln(0.6) = 0.0892.$$

Since $f(a)f(c) < 0$, set $a = a$ and $b = c$ such that $a = 0.2$, $b = 0.6$ and thus $c = (a + b)/2 = 0.8/2 = 0.4$.

The results for Example 15 is summarized in Table 2.

Table: Solution to Example 15 using Bisection method.

k	a	b	c	f(a)	f(b)	f(c)
1	0.2000	1.0000	0.6000	-1.4094	1.0000	0.0892
2	0.2000	0.6000	0.4000	-1.4094	0.0892	-0.5163
3	0.4000	0.6000	0.5000	-0.5163	0.0892	-0.1931
4	0.5000	0.6000	0.5500	-0.1931	0.0892	-0.0478
5	0.5500	0.6000	0.5750	-0.0478	0.0892	0.0216
6	0.5500	0.5750	0.5625	-0.0478	0.0216	-0.0129
7	0.5625	0.5750	0.5687	-0.0129	0.0216	0.0044

Since $|f(c)| = 0.0044$ and $0.0044 < \epsilon = 0.0050$, thus the iteration is stopped at the 7th iteration. The root of $x + \ln x$ is 0.5687. The graph of $f(x) = x + \ln(x)$ is shown below. As shown in Figure 2 when $y = 0$, the root is closer to 0.566, which should be similar to the result obtained above.

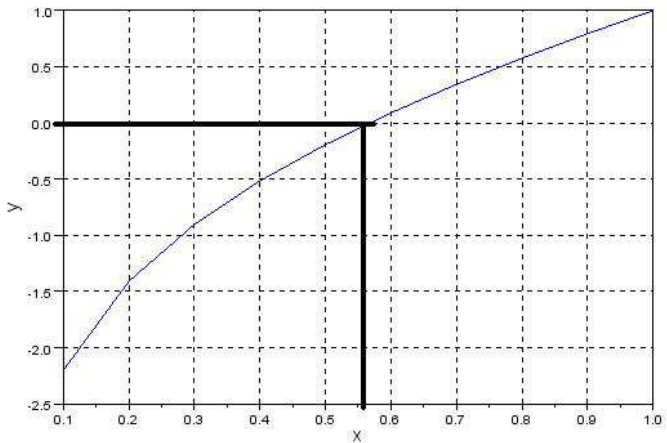


Figure: Graph of $x + \ln x$

Class Exercise

Find the root of

$$f(x) = \sin(2x) + x^3 - 3,$$

using Bisection method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[1, 2]$.

Answer: 1.379.

Class Exercise

Find the root of

$$f(x) = e^{-x} \cos(x),$$

using Bisection method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[-2, -1]$.

Answer: -1.298.

Class Exercise

Find the root of

$$f(x) = x^2 - 7,$$

using Bisection method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[2, 3]$. State if there is any other root?

Answer: 2.645.

Secant Method

In this method, a line through $(x^{(k-1)}, f_{k-1})$ and $(x^{(k-2)}, f_{k-2})$, where $f_{k-1} = f(x^{(k-1)})$ and $f_{k-2} = f(x^{(k-2)})$ is required.

This line is

$$y - f_{k-1} = \frac{f_{k-1} - f_{k-2}}{x^{(k-1)} - x^{(k-2)}} (x - x^{(k-1)}).$$

Taking $x^{(k)}$ as the x value where this crosses the x -axis (i.e. $y = 0$) leads to the **Secant equation** given as follows:

$$x^{(k)} = x^{(k-1)} - \frac{x^{(k-1)} - x^{(k-2)}}{f(x^{(k-1)}) - f(x^{(k-2)})} f(x^{(k-1)}).$$

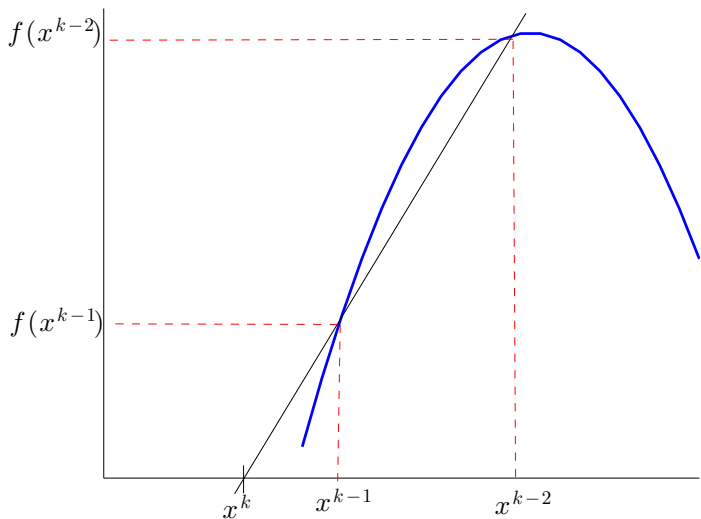


Figure: Secant method

Step by Step Guide to Secant Method

- 1 Need two initial values $[x^{(0)}, x^{(1)}]$.
- 2 Find the value of $f(x^{(1)})$ using $x^{(1)}$ and the value of $f(x^{(0)})$ using $x^{(0)}$.
- 3 Find the value of $x^{(2)}$ for $k = 2$ using

$$x^{(k)} = x^{(k-1)} - \frac{x^{(k-1)} - x^{(k-2)}}{f(x^{(k-1)}) - f(x^{(k-2)})} f(x^{(k-1)}), \quad k = 2, 3, \dots$$

- 4 Using the formula above we can find the value of $x^{(3)}, f(x^{(3)}), x^{(4)}, f(x^{(4)})$ and etc.
- 5 Repeat until we obtain $f(x^{(k)}) = 0$ or $|f(x^{(k)})| < \epsilon$.

Secant Method

Example

Apply two iterations of the Secant method to the equation $x^2 = 0$.
Use $x^{(0)} = 1$ and $x^{(1)} = 0.5$.

Solution

$$\begin{aligned}x^{(2)} &= x^{(1)} - \frac{x^{(1)} - x^{(0)}}{f(x^{(1)}) - f(x^{(0)})} f(x^{(1)}), \\&= 0.5 - \frac{0.5 - 1}{0.5^2 - 1^2} 0.5^2, \\&= 0.5 - \frac{-0.5}{-0.75} 0.5^2, \\&= 0.3333.\end{aligned}$$

and

$$\begin{aligned}x^{(3)} &= x^{(2)} - \frac{x^{(2)} - x^{(1)}}{f(x^{(2)}) - f(x^{(1)})} f(x^{(2)}), \\&= 0.3333 - \frac{0.3333 - 0.5}{0.3333^2 - 0.5^2} 0.3333^2, \\&= 0.2000.\end{aligned}$$

Example

Find the root of $x + \ln(x) = 0$ over the interval $[0.2, 1]$ by using secant method. Iterate until $|f(x_k)| < \epsilon = 0.005$.

Solution

$$\begin{aligned}x^{(2)} &= x^{(1)} - \frac{x^{(1)} - x^{(0)}}{f(x^{(1)}) - f(x^{(0)})} f(x^{(1)}), \\ &= 1 - \frac{1 - 0.2}{(1 + \ln(1)) - (0.2 + \ln(0.2))} (1 + \ln(1)), \\ &= 1 - \frac{0.8}{1 - (-1.4094)} (1) = 0.6680.\end{aligned}$$

Solution

$$\begin{aligned}x^{(3)} &= x^{(2)} - \frac{x^{(2)} - x^{(1)}}{f(x^{(2)}) - f(x^{(1)})} f(x^{(2)}), \\ &= 0.668 - \frac{0.668 - 1}{(0.668 + \ln(0.668)) - (1 + \ln(1))} (0.668 + \ln(0.668)), \\ &= 0.668 - \frac{-0.332}{0.2645 - 1} (0.2645) = 0.549.\end{aligned}$$

Next, calculate $x^{(4)}$. However, it is easier to check the result of $|f(x_i)| < \epsilon = 0.005$ by producing a table as given in Table 3.

Table: Solution to Example 19 for $x + \ln(x) = 0$ using Secant method

k	$x^{(k)}$	$f(x^{(k)})$
0	0.2	-1.4094
1	1	1.000
2	0.668	0.2645
3	0.549	- 0.0507
4	0.568	0.0020

Since $f(x^{(4)}) = 0.002 < \epsilon = 0.0050$, therefore the root of $x + \ln(x)$ is 0.5680.

It can be observed that in Example 15 that Bisection method requires 7 iterations to satisfy the stopping criteria $|f(x_k)| < \epsilon = 0.005$ while Secant method only requires 4 iterations.

This results show that Secant method converge faster than Bisection method.

Class Exercise

Find the root of

$$f(x) = \sin(2x) + x^3 - 3,$$

using Secant method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[1, 2]$.

Answer: 1.379.

Class Exercise

Find the root of

$$f(x) = e^{-x} \cos(x),$$

using Secant method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[-2, -1]$.

Answer: -1.298.

Class Exercise

Find the root of

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Answer: 2.645.

Newton method

From the knowledge of Calculus, slope of the tangent is $f'(x^{(k-1)})$.
The formula for the slope is given by

$$m = \frac{f(x^{(k-1)})}{x^{(k-1)} - x^{(k)}}.$$

Re-arranging the equations gives,

$$\begin{aligned} f'(x^{(k-1)}) &= \frac{f(x^{(k-1)})}{x^{(k-1)} - x^{(k)}}, \\ x^{(k)} &= x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})}. \end{aligned}$$

This can be written as

$$x^{(k)} = x^{(k-1)} - \frac{f_{k-1}}{f'_{k-1}}.$$

This equation is often referred to as the Newton equation and $\frac{f_{k-1}}{f'_{k-1}}$ as the *Newton update* or *Newton correction*.

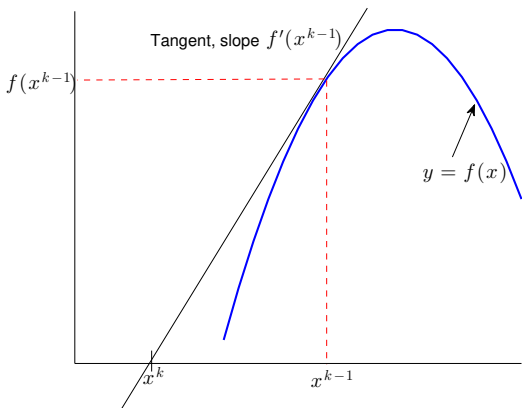


Figure: Newton-Raphson method

Newton Method

Example

Write down the Newton equation for $f(x) = \cos(x) + 0.6$.

Solution

We have $f'(x) = -\sin(x)$. Hence the Newton equation is

$$\begin{aligned}x^{(k)} &= x^{(k-1)} - \frac{\cos(x^{(k-1)}) + 0.6}{-\sin(x^{(k-1)})}, \\ &= x^{(k-1)} + \frac{\cos(x^{(k-1)}) + 0.6}{\sin(x^{(k-1)})}.\end{aligned}$$

Step by Step Guide to Newton method

- 1 Need one initial value $x^{(0)}$.
- 2 Find $f'(x^{(k-1)})$.
- 3 Find the value of $x^{(1)}$ for $k = 1$ using

$$x^{(k)} = x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})}.$$

- 4 Using the formula above we can find the value of $x^{(3)}, x^{(4)}$ and etc.
- 5 Repeat until we obtain $f(x^{(k)}) = 0$ or $|f(x^{(k)})| < \epsilon$.

Example

Find the root of $x + \ln x = 0$ with $x^{(0)} = 1$ by using Newton Raphson method. Iterate until $|f(x^k)| < \epsilon = 0.005$.

Solution

Find the value of $f'(x^{(k-1)})$ when $x^{(k-1)}$ is given as $x^{(0)} = 1$.

$$f(x) = x + \ln x, \quad f(x^{(0)}) = f(1) = 1 + \ln 1 = 1.$$

$$f'(x) = 1 + \frac{1}{x}, \quad f'(x^{(0)}) = f'(1) = 1 + 1/1 = 2.$$

Now using the Newton-Raphson formula, find all the x^k values for $k = 1, 2, 3 \dots$ until $|f(x_k)| < \epsilon = 0.005$.

$$x^{(k)} = x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})}.$$

Solution*Iteration 1, $k = 1$*

$$\begin{aligned}x^{(1)} &= x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}, \\ &= 1 - \frac{1}{2} = 0.5000.\end{aligned}$$

$$\begin{aligned}f(x) &= x + \ln x, & f(x^{(1)}) &= f(0.5) = 0.5 + \ln 0.5 = -0.1931, \\ f'(x) &= 1 + \frac{1}{x}, & f'(x^{(1)}) &= f'(0.5) = 1 + 1/0.5 = 3.0000.\end{aligned}$$

Solution*Iteration 2, $k = 2$*

$$\begin{aligned}x^{(2)} &= x^{(1)} - \frac{f(x^{(1)})}{f'(x^{(1)})}, \\ &= 0.5 - \frac{-0.1931}{2.5000} = 0.5643.\end{aligned}$$

$$f(x) = x + \ln x, \quad f(x^{(2)}) = f(0.5643) = 0.5643 + \ln 0.5643 = -0.0079,$$

$$f'(x) = 1 + \frac{1}{x}, \quad f'(x^{(2)}) = f'(0.5643) = 1 + 1/0.5643 = 2.7721.$$

*Solution**Iteration 3, $k = 3$*

$$\begin{aligned} x^{(3)} &= x^{(2)} - \frac{f(x^{(2)})}{f'(x^{(2)})}, \\ &= 0.5643 - \frac{-0.0079}{2.7721} = 0.5671. \end{aligned}$$

$$f(x) = x + \ln x, \quad f(x^{(3)}) = f(0.5671) = 0.5671 + \ln 0.5671 = -0.00012$$

$$f'(x) = 1 + \frac{1}{x}, \quad f'(x^{(3)}) = f'(0.5671) = 1 + 1/0.5671 = 2.7634.$$

Since $|f(x^{(3)})| = 0.0001196 < \epsilon = 0.005$, the calculations is stopped. Hence the root of $x + \ln x$ is $\approx x^{(3)} = 0.5671$.

The answer to this iterations is summarized in Table 4.

Table: Solution to Example 24 for $x + \ln(x) = 0$ using Newton method

k	$x^{(k)}$	$f(x^{(k)})$	$f'(x^{(k)})$
0	1	1	2
1	0.5000	-0.1931	3.0000
2	0.5643	-0.0079	2.7721
3	0.5671	-0.00012	2.7634

Comparing with Example 15 and Example 19, Newton Raphson method only requires 3 iterations to converge.

Thus, Newton-Raphson converges faster than Secant and Bisection methods.

Example

Given the graph of the function $f(x) = \sin 2x + x^3 - 3$. Find the root of $f(x)$ by using Newton-Raphson method. Use the initial estimate $x^{(0)} = 1$. Iterate until $|f(x^k)| < \epsilon = 0.005$.

Solution

We have

$$f(x) = \sin(2x) + x^3 - 3,$$

$$f(x^{(0)}) = f(1) = \sin(2(1)) + (1)^3 - 3 = -1.0907.$$

$$f'(x) = 2 \cos(2x) + 3x^2,$$

$$f'(x^{(0)}) = f'(1) = 2 \cos(2(1)) + 3(1)^2 = 2.1677.$$

Solution

Iteration 1, $k = 1$

$$\begin{aligned}x^{(1)} &= x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}, \\ &= 1 - \frac{-1.0907}{2.1677} = 1.5032.\end{aligned}$$

We have $x^{(1)} = 1.5032$.

$$f(x) = \sin(2x) + x^3 - 3,$$

$$f(x^{(1)}) = f(1.5032) = \sin(2(1.5032)) + (1.5032)^3 - 3 = 0.5314.$$

$$f'(x) = 2 \cos(2x) + 3x^2,$$

$$f'(x^{(1)}) = f'(1.5032) = 2 \cos(2(1.5032)) + 3(1.5032)^2 = 4.7971.$$

Solution

Iteration 2, $k = 2$

$$\begin{aligned}x^{(2)} &= x^{(1)} - \frac{f(x^{(1)})}{f'(x^{(1)})}, \\ &= 1.5032 - \frac{0.5314}{4.7971} = 1.3924.\end{aligned}$$

We have $x^{(2)} = 1.3924$.

$$f(x) = \sin(2x) + x^3 - 3,$$

$$f(x^{(2)}) = f(1.3924) = \sin(2(1.3924)) + (1.3924)^3 - 3 = 0.0488.$$

$$f'(x) = 2 \cos(2x) + 3x^2,$$

$$f'(x^{(2)}) = f'(1.3924) = 2 \cos(2(1.3924)) + 3(1.3924)^2 = 3.9423.$$

Solution*Iteration 3, $k = 3$*

$$\begin{aligned}x^{(3)} &= x^{(2)} - \frac{f(x^{(2)})}{f'(x^{(2)})}, \\ &= 1.3924 - \frac{0.0488}{3.9423} = 1.3800.\end{aligned}$$

Solution

We have $x^{(3)} = 1.3800$.

$$f(x) = \sin(2x) + x^3 - 3,$$

$$f(x^{(3)}) = f(1.3800) = \sin(2(1.3800)) + (1.3800)^3 - 3 = 0.000471.$$

$$f'(x) = 2 \cos(2x) + 3x^2,$$

$$f'(x^{(3)}) = f'(1.3800) = 2 \cos(2(1.3800)) + 3(1.3800)^2 = 3.8571.$$

Since $|f(x^{(3)})| = 0.000471 < \epsilon = 0.005$, the calculations is stopped. Hence the root of $\sin(2x) + x^3 - 3$ is $\approx x^{(3)} = 1.3800$.

The answer to this iterations is summarized in Table 5.

Table: Solution to Example 29 for $f(x) = \sin 2x + x^3 - 3$ using Newton method

k	$x^{(k)}$	$f(x^{(k)})$	$f'(x^{(k)})$
0	1	- 1.0907	2.1677
1	1.5032	0.5314	4.7971
2	1.3924	0.0488	3.9423
3	1.3800	0.000471	3.8571

Class Exercise

Find the root of

$$f(x) = \sin(2x) + x^3 - 3,$$

using Newton method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[1, 2]$.

Answer: 1.379.

Class Exercise

Find the root of

$$f(x) = e^{-x} \cos(x),$$

using Newton method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[-2, -1]$.

Answer: -1.298.

Class Exercise

Find the root of

$$f(x) = x^2 - 7,$$

using Newton method. Iterate until $|f(c)| < \epsilon = 0.005$ over the interval $[2, 3]$. State if there is any other root?

Answer: 2.645.

Convergence

The convergence of the three iterative methods are discussed in this section. It is sufficient for this course to think of convergence as meaning

$$\lim_{k \rightarrow \infty} x^{(k)} = x^*.$$

Local Convergence

Definition

A method is said to converge locally if there is a neighbourhood of the root x^* such that for all $x^{(0)}$ in the neighbourhood, the iterates $x^{(0)}, x^{(1)}, x^{(2)}, \dots$, converge to the root x^* .

Global Convergence

Definition

A method is said to converge globally if the iterates $x^{(0)}, x^{(1)}, x^{(2)}, \dots$, converge to the root x^* for all $x^{(0)}$.

Rate of Convergence

Define the error by $e^{(k)} := x^{(k)} - x^*$. A method is said to converge with a rate of convergence at least p if

$$|e^{(k)}| = C_p |e^{(k-1)}|^p \quad \text{as } k \rightarrow \infty,$$

where C_p is a constant, often called the *asymptotic error constant*.

Method	Order p	Convergence
Newton	2 (quadratic)	Local convergence
Secant	1.1618 (super linear)	Local convergence
Bisection	1 (linear)	Global convergence

Summary of the iterative methods

Bisection Method

The simplest root-finding algorithm is the bisection method. It works when f is a continuous function and it requires previous knowledge of two initial guesses, a and b , such that $f(a)$ and $f(b)$ have opposite signs. Although it is reliable, it converges slowly, gaining one bit of accuracy with each iteration. The order p is 1 and the convergence is **linear**.

Summary of the iterative methods

Secant Method

Replacing the derivative in Newton's method with a finite difference, we get the secant method. This method does not require the computation (nor the existence) of a derivative, but the price is slower convergence. The order p is approximately 1.6 and the convergence is **super linear**.

Summary of the iterative methods

Newton Method

Newton's method assumes the function f to have a continuous derivative. Newton's method may not converge if started too far away from a root. However, when it does converge, it is faster than the bisection method. The order p is 2 and the convergence is **quadratic**. Newton's method is also important because it readily generalizes to higher-dimensional problems.